# Filtered Leapfrog Time Integration with Enhanced Stability Properties


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## Initial Value Problem

- Our goal is to solve the initial value problem

$$y' = F(t, y(t)); \qquad y(0) = y^0$$

by numerical approximation using the Leapfrog Method

## Linear Multistep Method: Leapfrog

- A general form for s-step linear multi-step methods will have the form

$$\sum_{m=0}^{s} \alpha_m y^{n+m} = k \sum_{m=0}^{s} \beta_m F[t^{n+m}, y^{n+m}], n = 0, 1, ...$$

- Other methods, such as Euler's method, must be used to calculate the first $s-1$ steps
- The **Leapfrog Method** is given by the formula

$$y^{n+1} = y^{n-1} + 2kF(t^n, y^n)$$

- Well known for its applications in science, engineering, weather forecasting, etc

## Stability of Leapfrog Method

- Stability is concerned with preventing errors in previous steps from being magnified in subsequent calculations

- The absolute stability region for Leapfrog Method is only the interval $[-i, i]$ in the complex plane

- This limits applicability and correctness [Figure 1]

- We use filtering to expand the absolute stability region

- We consider a filter of the form

$$P(E) = \sum_{j=-r_1}^{r_2} a_j E^j$$

[Aluthge]

- E is the forward shift defined by $Ey(t) = y(t+k)$, where $k$ is the step length

## Construction of the Filter

- We will construct a five-point, symmetric filter with $r_1, r_2 = 2$, and thus

$$P_5^{(0)}(E) = a_{-2}E^{-2} + a_{-1}E^{-1} + a_0E^0 + a_1E^1 + a_2E^2$$

- We construct the filter with the condition $P(e^s) = 1 + \mathcal{O}(s^3)$ and $P(-e^{-s}) = \mathcal{O}(s^2$ and solve the resulting linear system to obtain the filter

$$P_5^{(0)}(E) = \frac{1}{16}(-E^{-2} + 4E^{-1} + 10E^0 + 4E^1 - E^2)$$

## New Linear Multistep Method

- The $P_5^{(0)}$ filter can be incorporated resulting in the new LMM

$$y^{n+1} = \frac{1}{16}(-y^{n-3} + 4y^{n-2} + 10y^{n-1} + 4y^n - [y^{n-1} + 2kF(t^n, y^n)])$$

$$+2kF(t^n, y^n)$$

- This method is zero-stable, while still maintaining the second order accuracy of the leapfrog method.

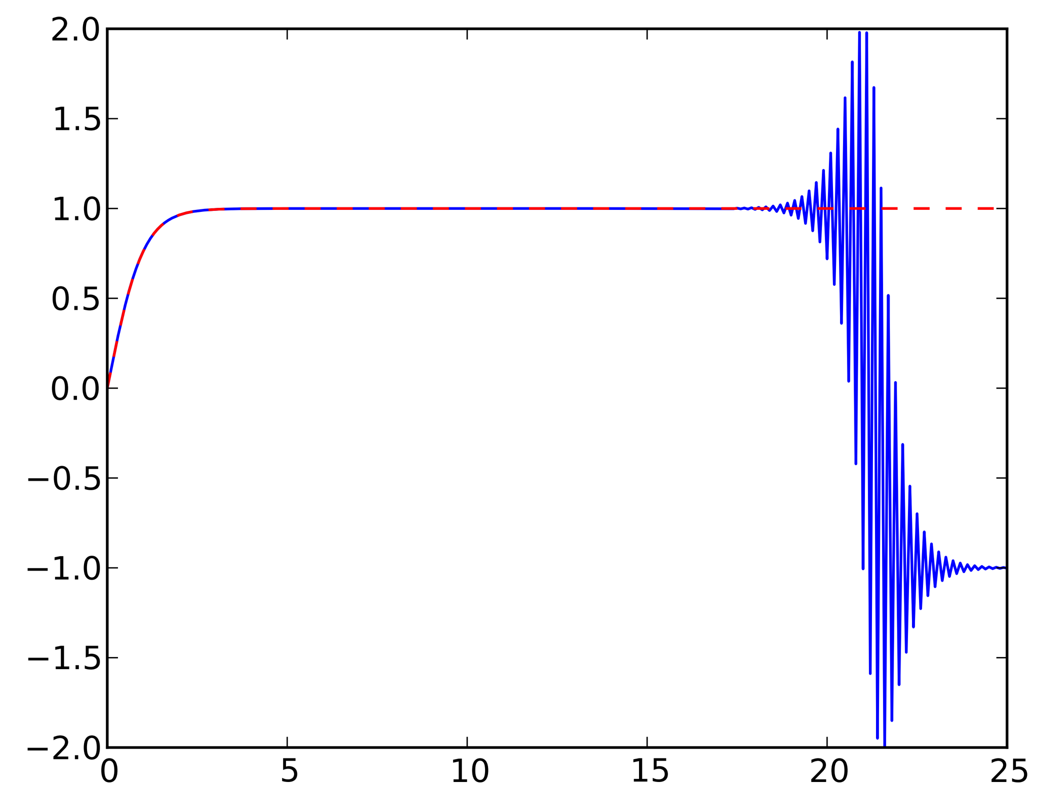


Figure 1: The real solution of our IVP (red, dashed) versus our computed solution under Leapfrog (blue, solid)

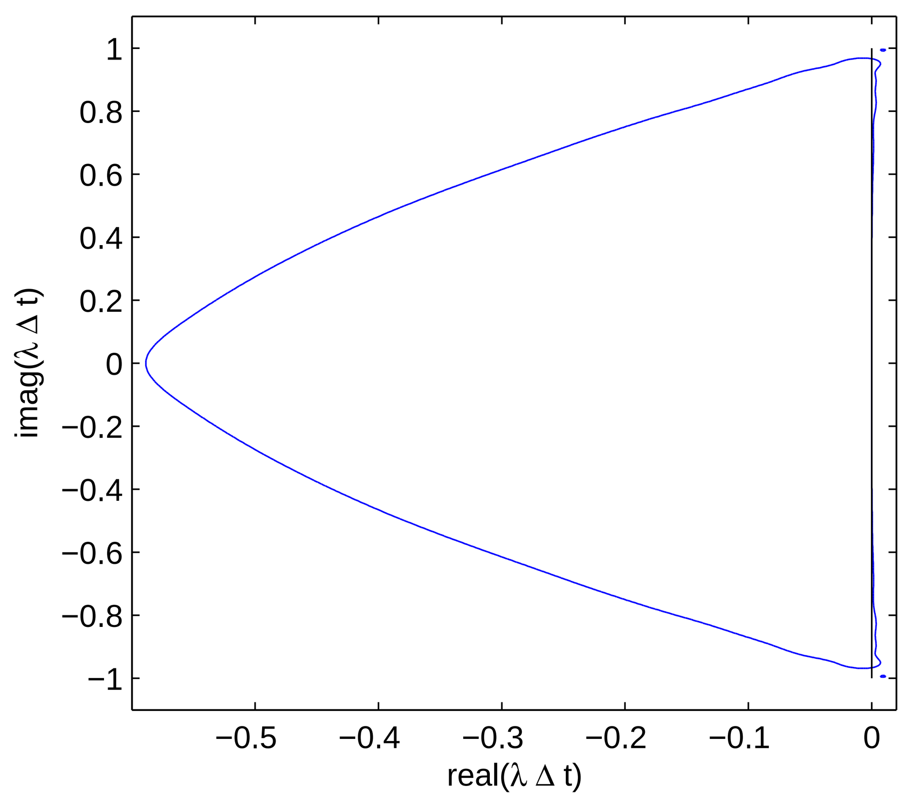


Figure 2: The Absolute Stability Region under the filter

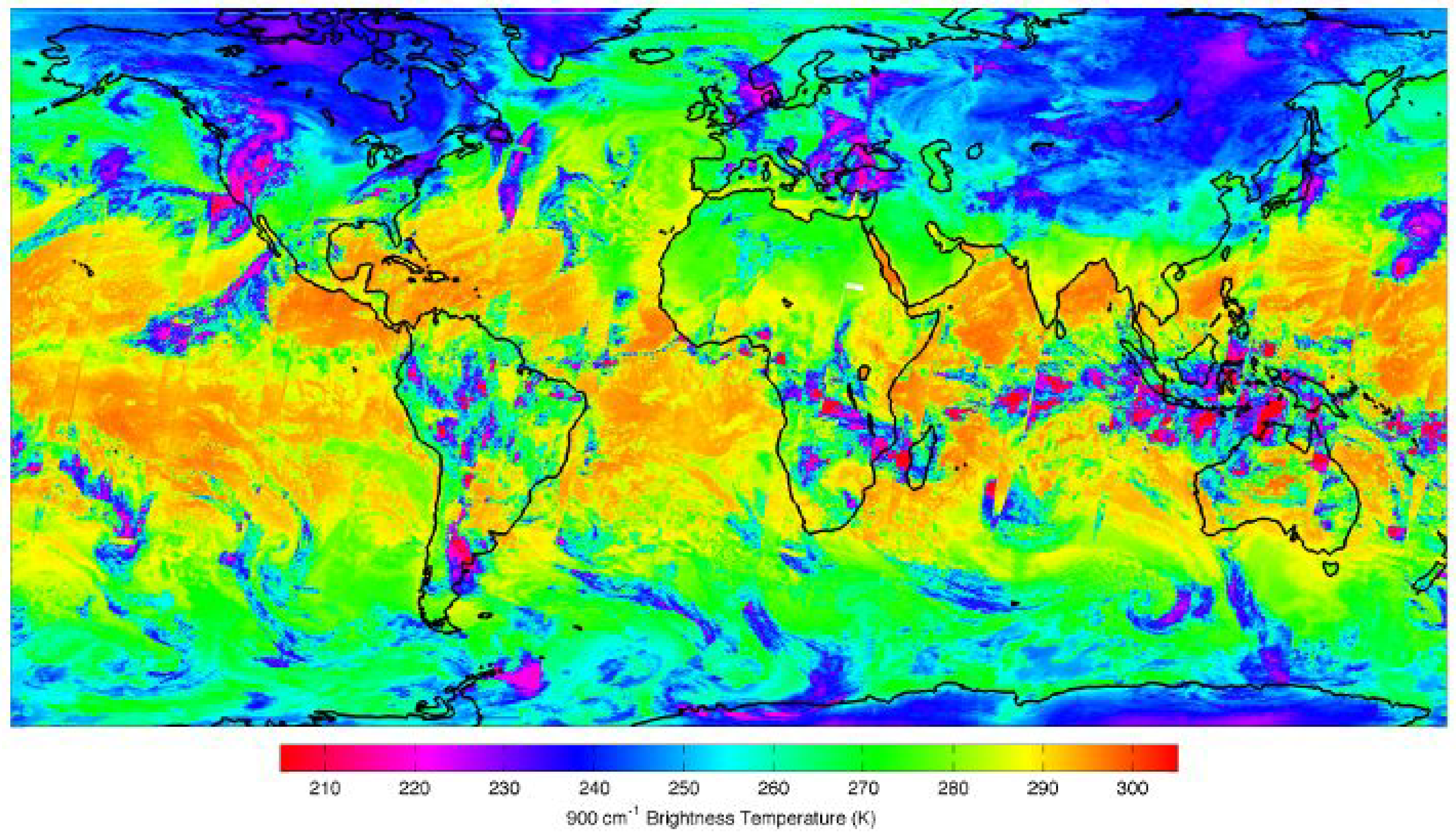


Figure 3: A graphic example of weather forecasting

## Application of the Filter

- To advance from the initial condition $y^0$ to the first time step $y^1$, $M$ smaller substeps are taken using $M$ Euler steps and $M-1$ Leapfrog Steps.

- Each step $y^n$ is replaced with an average of $y^n$ with the neighboring time levels
- This process is restarted after $C$ timestep evaluations, at which $y^C$ becomes the initial condition

- Resulting absolute stability region covers $[-.95i, .95i]$, but expands horizontally within the complex plane [Figure 2]

## Numerical Examples

| | $t=5$ | $t=25$ | $t=100$ |
|---|---|---|---|
| leapfrog | 4.2e-6 | 3.0 | $\infty$ |
| $P_3^{(0)}$ LMM | 2.9e-5 | 0 | 0 |
| $P_5^{(0)}$ LMM | 4.2e-6 | 3.3e-16 | 3.3e-16 |
| M1 | 1.3e-6 | 2.2e-16 | 2.2e-16 |
| M2 | 7.8e-5 | 0 | 0 |
| M3 | 1.7e-7 | 0 | 0 |
| M4 | 2.6e-6 | 0 | 0 |
| Asym $P_5^{(-2)}$ M2 | 4.5e-5 | 2.2e-16 | 2.2e-16 |

Note: We define M1 as the method of filtering after every step,
M2 as the process of filtering after every $N$ steps,
M3 as the process of taking smaller, Euler substeps to reach the first step,
then proceeding like M2, M4 as the method discussed in the Application of Filter section

## Application

The Leapfrog Method is considered an important tool for numerical weather forecasting [Coiffier]. However, the limited absolute stability region can cause important error for eigenvalues that do not fall in this narrow strip.
The described method is a potential replacement for existing leapfrog type time differencing schemes that are used in this style of weather forecasting.

## Bibliography


- A. Aluthge (aka A. Ariyadasa). Filtering and extrapolation techniques in the numerical solution of ordinary differential equations. Masters thesis, University of Ottawa, 1985.
- J. Coiffier. Fundamentals of Numerical Weather Prediction. Cambridge University Press, 2011.