Leapfrog Method Time Integration in Numerical Weather Models

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Introduction to Weather Modeling

Numerical weather prediction (NWP) is a concept that relies heavily on computational mathematics. NWP uses numerical mathematics to predict future atmospheric conditions. These systems of differential equations, representing physical conditions, often cannot be solved analytically, so numerical methods are used. The difficulty is in balancing speed and accuracy when implementing these methods.

Leapfrog Method in NWP

The leapfrog integration scheme is a popular method in NWP with regards to time integration. NWP uses leapfrog because it provides second-order accuracy while only requiring the storage of one previous time step. Because NWP needs to evaluate massive numbers of points, memory storage and speed are important to take into consideration.

Filtering Techniques

Numerical methods, such as leapfrog, have various reasons that can cause them to go unstable. The most common way to determine the stability region is to look at the eigenvalues (λ) and time-step size (Δt). The product, $\lambda\Delta t$, can be plotted using a *root locus curve*, which shows the boundaries of the area of stability for that technique (Figure c).

The leapfrog method's stability region from the root locus curve is only the imaginary axis from $[-i, i]$. In practice, $\lambda\Delta t$ with values in $\Re: [-1, 0], \Im: [-i, i]$ will be used. Filtering techniques perform additional calculations to expand the stability region to include more of these values of $\lambda\Delta t$.

Leapfrog Filters in Use

- The Global Forecast System (GFS) Global Spectrum Model (GSM)
- Models used by the meteorological agencies of Japan, Australia, Germany, and Netherlands
- General atmospheric circulation models
- General ocean circulation models

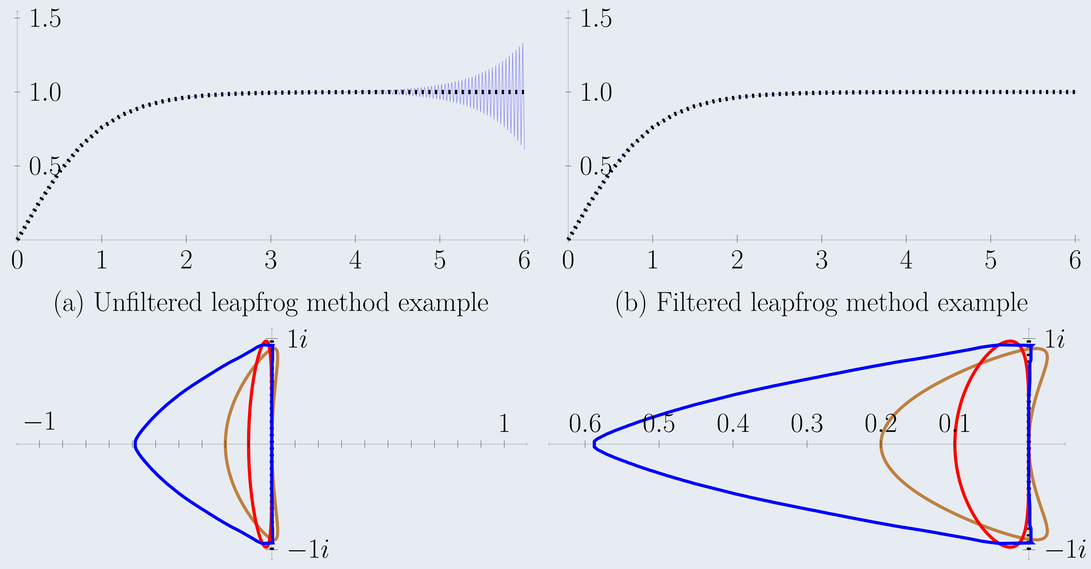
Leapfrog Method

The leapfrog method is derived from the central difference method to explicitly solve for $f(t + \Delta t)$,

$$f(t + \Delta t) = f(t - \Delta t) + 2\Delta t F[f(t)], \text{ where} f' = F[f(t)].$$

One major problem with the leapfrog method, though, is that it has a very small region of stability, and the computational error can overwhelm the physical solution, as shown in the figure.

Filtering Techniques in Action



(a) Unfiltered leapfrog method example

(b) Filtered leapfrog method example

(c) Left: The stability regions of unfiltered (black, dotted), Robert-Asselin filtering (brown), Robert-Asselin-Williams filtering (red), and Aluthge-Sarra filtering (blue); Right: Zoomed-in version

Robert-Asselin Filter

One of the most widespread filtering techniques was developed by Robert and Asselin in the 1970s. This technique uses the regular leapfrog method, then applies the filter

$$\overline{F(t)} = F(t) + 0.5v\big[\overline{F(t - \Delta t)} - 2F(t) + F(t + \Delta t)\big].$$

This technique has been widely implemented, but it does have some weaknesses - it reduces the leapfrog method to first order accuracy and can even distort the physics being modeled.

Robert-Asselin-Williams Filter

Because of the drawbacks inherent in the Robert-Asselin filter, in 2009, Williams proposed a slight modification to the technique that could be easily implemented. He states that his modification preserves the physics and increases accuracy to third-order. The leapfrog method is filtered twice, with

$$\overline{\overline{F}}(t) = \overline{F}(t) + 0.5v\alpha\big[\overline{\overline{F}}(t - \Delta t) - 2\overline{F}(t) + F(t + \Delta t)\big],$$
$$\overline{F}(t + \Delta t) = F(t + \Delta t) - 0.5v(1 - \alpha)\big[\overline{\overline{F}}(t - \Delta t) - 2\overline{F}(t) + F(t + \Delta t)\big].$$

Aluthge-Sarra Filter

Dr. Aluthge and Dr. Sarra of Marshall University proposed a filtering technique that further expands the stability region of the leapfrog method.

The filter, shown in the third step below, was developed by Dr. Aluthge, and the technique to use it in this way was designed by Dr. Sarra.

1. The method is initialized with a series of M substeps between f_0 and f_1. The first of these substeps implements Euler's method. The other $M - 1$ implement the ordinary leapfrog method.
2. With f_0 and f_1 now defined, the next N time steps are found by the unfiltered leapfrog method.
3. The Nth and $(N - 1)$th steps are then filtered using Dr. Aluthge's 5-point symmetric filter, defined as
$$f_{n+1} = \frac{1}{4}f_n + \frac{9}{16}f_{n-1} + \frac{1}{4}f_{n-2} - \frac{1}{16}f_{n-3} + \frac{15}{8}kF[y_n].$$
4. This process in Step 3 is repeated C times.
5. At the end of Step 4, the process is restarted from Step 1, with the last (filtered) result acting as the new initial condition, "f_0".

This technique is shown in their paper to include almost the entirety of the imaginary axis in its stability region, as the unfiltered leapfrog method does, as well as much of the region that covers equations that require filtering of the computational mode.

References

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