A Pseudospectral Method with Edge Detection Free Postprocessing for 2d Hyperbolic Heat Transfer*

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Abstract

Under the governing equations of Hyperbolic Heat Transfer thermal disturbances travel with a finite speed of propagation and are visible as sharp discontinuities in the solution profiles. Due to the well-known Gibbs phenomenon, the numerical solution of hyperbolic heat transfer problems by high order numerical methods such as pseudospectral methods will feature non-physical numerical oscillations. For pseudospectral methods, postprocessing methods have been developed to lessen or even eliminate the effects of the Gibbs phenomenon. The most powerful postprocessing methods require that the exact location of the discontinuities be known. Due to the reflection and interaction of thermal waves, the solutions of multi-dimensional hyperbolic heat transfer problems often have very complex features which make accurately locating discontinuities difficult or even impossible. The application of an edge detection free postprocessing method which is effective for this type of problem is discussed.

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NOMENCLATURE

a	DTV postprocessing regularization parameter
a_k	Chebyshev expansion coefficients
D	Chebyshev differentiation matrix
N	N+1 collocation points
N_α	DTV postprocessing neighborhood
S	dimensionless energy generation rate
T	dimensionless temperature
T_k	k^{th} order Chebyshev polynomial
t	dimensionless time
Q	dimensionless heat flux in the x direction
R	dimensionless heat flux in the y direction
x	dimensionless space variable
y	dimensionless space variable
γ	coordinate map parameter
λ	DTV postprocessing fitting parameter
$\lvert\nabla_\alpha u\rvert_a$	DTV postprocessing strength function

1 Introduction

In situations where heat transfer occurs over a very short period of time at very high or very low temperatures, the classical diffusion (parabolic) theory of heat transfer breaks down since the wave nature of thermal energy transport becomes dominant. The hyperbolic heat equations model this process and it results in energy propagating through a medium as a wave with sharp discontinuities at the wave front.

In one dimension, the dimensionless governing equations of Hyperbolic Heat Transfer are

$$\begin{aligned} T_t + Q_x &= S/2 \\ Q_t + T_x &= -2Q \end{aligned} \tag{1}$$

where $T(x,t)$ is the temperature, $Q(x,t)$ is the heat flux, and $S(x,t)$ is the energy generation rate. In two space dimensions the equations become

$$\begin{aligned} T_t + Q_x + R_y &= S/2 \\ Q_t + T_x &= -2Q \\ R_t + T_y &= -2R \end{aligned} \tag{2}$$

where $T(x,y,t)$ is the temperature, $Q(x,y,t)$ is the heat flux in the x direction, $R(x,y,t)$ is the heat flux in the y direction, and $S(x,y,t)$ is the energy generation rate.

Previously, various numerical methods have been used to obtain approximate solutions of hyperbolic heat transfer problems. First, local methods such as second-order finite difference methods were applied as in references [1, 2, 3]. With finite difference methods often as many as 1000 grid points were used in a small one-dimensional domain to resolve the problem and oscillations were still visible around the discontinuity at the temperature front. More recently, several authors [4, 5] have used non-oscillatory second-order finite difference methods which suppress oscillations by using a flux or slope limiter.

The superiority of pseudospectral methods over lower order local methods for the solution of partial differential equations with sufficiently smooth solutions has been well established. When the solution is sufficiently smooth pseudospectral methods exhibit what is termed spectral or exponential accuracy. However, when solutions are only piecewise smooth the Gibbs phenomenon appears as an accuracy reduction to first order away from discontinuities and $\mathcal{O}(1)$ oscillations in the neighborhoods of jumps. Despite producing oscillatory solutions on problems with discontinuities, pseudospectral methods still have a huge advantage over lower order finite difference methods since they only have minimal dispersion and dissipation errors and are able to match the accuracy of finite difference methods with significantly fewer grid points.

Over the last decade or so, several postprocessing methods have been developed to lessen or even completely remove the effects of the Gibbs phenomenon. A survey of pseudospectral postprocessing methods and software that implements the methods can be found in [6]. In [7] we examined the application of the Chebyshev pseudospectral method with Gegenbauer reprojection postprocessing for hyperbolic heat conduction problems in one-dimension. Applications in two dimensions were not considered because, while theoretically possible, we were unable to apply the Gegenbauer reprojection method in two dimensions. The Gegenbauer reprojection method requires that exact locations of each discontinuity be precisely known and then two function dependent parameters must be specified in each smooth subinterval. The application of the Gegenbauer reprojection method may be difficult in one-dimension, but its application is nearly impossible for functions with complex features in two-dimensions. This is especially true if the discontinuities are not orthogonal to the cartesian grid.

The purpose of this work is to describe the solution of hyperbolic heat transfer problems in two-dimensions by the Chebyshev pseudospectral method and then describe an efficient, easy to implement, postprocessing algorithm that lessens the effects of the Gibbs phenomenon and sharply resolves steep fronts without smearing.

2 Chebyshev Pseudospectral Method

The Chebyshev Pseudospectral (CPS) method is based on the interpolating Chebyshev expansion

$$\mathcal{I}_N u(x) = \sum_{k=0}^{N} a_k \, T_k(x) \tag{3}$$

standardized to the interval $\Omega = [-1, 1]$. The Chebyshev polynomials are

$$T_k\,(x) = \cos(k \arccos\,(x)) \tag{4}$$

and the Chebyshev expansion coefficients are denoted as a_k. The expansion satisfies $\mathcal{I}_N u(x_j) = u(x_j)$ at $N+1$ interpolation cites x_j. The interpolation sites or collocation points are the Chebyshev-Gauss-Lobatto (GCL) points

$$x_j = -\cos\left(\frac{\pi j}{N}\right), \qquad j = 0, 1, ..., N \tag{5}$$

which cluster densely around the boundaries of the domain. In applications of the CPS method for solving time-dependent PDEs, it is common to redistribute the CGL points via the map [8]

$$x_j = \frac{\arcsin[-\gamma \cos(\pi j/N)]}{\arcsin(\gamma)}, \quad j = 0, 1, \ldots N, \quad 0 < \gamma < 1. \tag{6}$$

The mapped points are more uniformly spaced and allow a larger stable time step when using explicit time integration methods.

Space derivatives can be approximated by multiplying by the Chebyshev differentiation matrix D as

$$u_x = Du$$

in $\mathcal{O}(N^2)$ floating point operations. Formulas for the entries of D as well as more detailed information on the CPS method can be found in the standard references [9, 10, 11, 12, 13]. For large N, derivatives can efficiently be evaluated via the fast cosine transform in $\mathcal{O}(N \log N)$ floating point operations. Freely available software for implementing the CPS method is described in [14]. After the partial differential equation is discretized in space with the Chebyshev method, a method of lines approach is taken, and the system of ordinary differential equations

$$u_t = F(u, t)$$

is advanced in time with an ODE method. In the numerical examples we have used a fourth-order Runge-Kutta method. In higher dimensions, the CPS method is implemented on a tensor product grid that is a (mapped) CGL grid set up independently in each space direction.

3 Digital Total Variation Filtering

The Total Variation (TV) de-noising model is a popular image processing method to remove noise from a digital image. The model formulates a minimization problem which leads to a nonlinear Euler-Lagrange PDE to be solved by numerical PDE methods. In [15, 16] the authors develop a discrete version of the TV model on a graph and refer to it as Digital Total Variation (DTV) filtering. The DTV method was used to postprocess Chebyshev pseudospectral approximations of Hyperbolic Conservation Laws in [17]. The method works with point values in physical space and not with the spectral expansion coefficients. The DTV method does not need to know the location of edges. The point values may be located at scattered, non-structured cites, in complex geometries. The DTV method can be implemented in a very computationally efficient manner. While the method does mitigate the effects of the Gibbs phenomenon it does not make any claims of restoring spectral accuracy.

General points in the computational domain are denoted by $\alpha, \beta, \cdots$. The notation $\alpha \sim \beta$ indicates that α and β are neighbors. All the neighbors of a point α are denoted by

$$N_\alpha = \{\beta \in \Omega \,|\, \beta \sim \alpha\}. \tag{7}$$

In one dimension, N_α is simply the points to the left and right of the point being postprocessed. In two space dimensions there is more than one way to define N_α (figure 1). One is to consider at a point $\alpha_{i,j}$ four neighboring points, $N_\alpha^4 = \{\alpha_{i,j+1}, \alpha_{i+1,j}, \alpha_{i,j-1}, \alpha_{i-1,j}\}$ and another is an eight point neighborhood, $N_\alpha^8 = \{\alpha_{i,j+1}, \alpha_{i+1,j+1}, \alpha_{i+1,j}, \alpha_{i+1,j-1}, \alpha_{i,j-1}, \alpha_{i-1,j-1}, \alpha_{i-1,j}, \alpha_{i-1,j+1}\}$. We have used N_α^8 in the numerical examples, however the results with N_α^4 are similar.

The graph variational problem is to minimize the fitted TV energy

$$E_\lambda^{TV}(u) = \sum_{\alpha \in \Omega} |\nabla_\alpha u|_a + \frac{\lambda}{2} \sum_{\alpha \in \Omega} (u_\alpha - u_\alpha^0)^2 \tag{8}$$

where u^0 is the spectral approximation containing the Gibbs oscillations and λ the user specified fitting parameter. The unique solution to this problem is the solution of the nonlinear restoration equation

$$\sum_{\beta \sim \alpha} (u_\alpha - u_\beta) \left(\frac{1}{|\nabla_\alpha u|_a} + \frac{1}{|\nabla_\beta u|_a} \right) + \lambda (u_\alpha - u_\alpha^0) = 0 \tag{9}$$

where the regularized location variation or strength function at any point α is defined as

$$|\nabla_\alpha u|_a = \left[\sum_{\beta \in N_\alpha} (u_\beta - u_\alpha)^2 + a^2 \right]^{1/2}. \tag{10}$$

The regularization parameter a is a small (we have used $a = 0.0001$ in the numerical examples) value used to prevent a zero local variation and division by zero.

To solve the nonlinear system, time marching with the explicit Euler method is used to advance a preconditioned form of the (9)

$$\frac{du_\alpha}{dt} = \sum_{\beta \sim \alpha} (u_\alpha - u_\beta) \left(1 + \frac{|\nabla_\alpha u|_a}{|\nabla_\beta u|_a} \right) + \lambda \, |\nabla_\alpha u|_a \, (u_\alpha - u_\alpha^0). \tag{11}$$

to a steady state. Typically about 100 time steps are required. An effective stopping criteria for the time marching is for the relative L^1 residual between two consecutive time steps to be less than some tolerance, i.e.,

$$\frac{\left\| u^{[k+1]} - u^{[k]} \right\|_{L^1}}{\left\| u^{[k]} \right\|_{L^1}} \leq tol. \tag{12}$$

Equation (11) is essentially a local finite difference approximation with each time iteration taking $\mathcal{O}(N)$ floating point operations. Thus, the DTV filtering algorithm is computationally efficient as long as not too many time marching steps are taken.

An optimal value of the fitting parameter is not known. However, a large range of values for λ results in a "good" postprocessing. In general, stronger oscillations are best handled with a small fitting parameter (< 10) while weaker oscillations require a larger value of the fitting parameter. More details on selecting the value of the fitting parameter can be found in reference [17]. Freely available software that implements DTV filtering is described in [6]. Reference [6] also gives a more detailed comparison of the DTV filter with competing postprocessing methods.

4 Numerical Examples

To assess the accuracy of the method we first consider an example in one dimension with an exact solution. We take equations (1) with $S = 0$ and initial conditions $T(x, 0) = 0$, $Q(x, 0) = 0$, and boundary conditions $T(0, t) = 1$, $T(1, t) = 0$. An exact solution is given in reference [1]. The grid is set up with $N = 127$ and mapping parameter $\gamma = 0.99$. The solution is advanced in time with $\Delta t = 0.001$ to time $t = 0.5$. The oscillatory Chebyshev solution along with the exact temperature profile are in the left image of figure 2. DTV postprocessing is applied with fitting parameter $\lambda = 40$ and 200 time steps are taken to advance the solution to a steady state. The postprocessing took 0.03 seconds using the algorithm implemented in Matlab running on a desktop computer that is representative of current computer technology. The point-wise errors from the CPS method and the DTV postprocessed CPS method are compared in figure 3.

The first example in two dimensions considers equations (2) on a rectangular domain with dimensions $[0, 1] \times [0, 0.5]$ with initial conditions $T(x, y, 0) = 1$ and $Q(x, y, 0) = 0$. Boundary conditions $\frac{\partial T}{\partial x} = 0$ are enforced everywhere except where $x = 0$ and $0.125 \leq y \leq 0.375$ where a constant temperature of

$T = 2$ is maintained. The problem is solved with the Chebyshev pseudospectral method on a tensor product grid determined by $\gamma_x = 0.995$, $\gamma_y = 0.99$, $N_x = 200$, and $N_y = 100$. The solution is advanced to time $t = 0.6$ with $\Delta t = 0.00025$. The initial discontinuity in the temperature propagates to the right and expands until it reaches the top and bottom of the domain and then it is reflected. A contour plot of the Chebyshev pseudospectral solution is in the upper image of figure 4. The DTV postprocessed solution is in the lower image of figure 4. The postprocessing took less than one second. An exact solution to the problem is unknown. The results are in good qualitative agreement with the solution of the same problem by second-order total variation diminishing methods in [5] and [4].

The second two-dimensional example considers equations (2) on a square domain with dimensions $[0,1]\times[0,1]$. Initially, $Q(x,y,0) = 0$ and $T(x,y,0) = 0$ at all points except for in a circle of radius 0.01 centered at the point $(0.5, 0.5)$ where $T(x,y,0) = 1$. On the boundary of the square the temperature is fixed at $T = 0$. The problem is solved with the Chebyshev pseudospectral method on a tensor product grid determined by $\gamma_x = 0.999$, $\gamma_y = 0.999$, $N_x = 160$, and $N_y = 160$. A contour plot of the Chebyshev pseudospectral temperature solution and DTV postprocessed temperature at $t = 0.1$ are shown in the upper portion of figure 5. Postprocessing at 25,921 collocation points took less than one second. The lower portion of figure 5 shows the same information but at time $t = 1$ when the wavefront has reflected off the boundary and is traveling towards the origin. The solutions agree qualitatively with solution of the same problem by a second-order total variation diminishing method in [18].

5 Conclusions

The advantages of Pseudospectral methods over lower order finite difference and finite element methods in simulating wave type phenomena is well-documented. Pseudospectral methods have minimal phase and dissipation errors and can capture wave type phenomena using relatively few collocation points. The accuracy of pseudospectral methods is severely degraded by Gibbs oscillations when discontinuities are present. Several pseudospectral postprocessing methods have been developed to lessen or even eliminate the effects of the Gibbs phenomenon. The most powerful postprocessing methods require that the exact location of the discontinuities be known. The complex features in the solutions of multi-dimensional hyperbolic heat transfer problems make it difficult to accurately locate discontinuities and methods that depend on the information perform poorly or fail. Additionally, many such postprocessing methods require one or more function dependent parameters to be specified in each smooth subinterval while the DTV method only requires one global parameter. In this work, DTV filtering has been shown to be an effective algorithm for efficiently lessening the effects of the Gibbs phenomenon in two-dimensional hyperbolic heat transfer problems.

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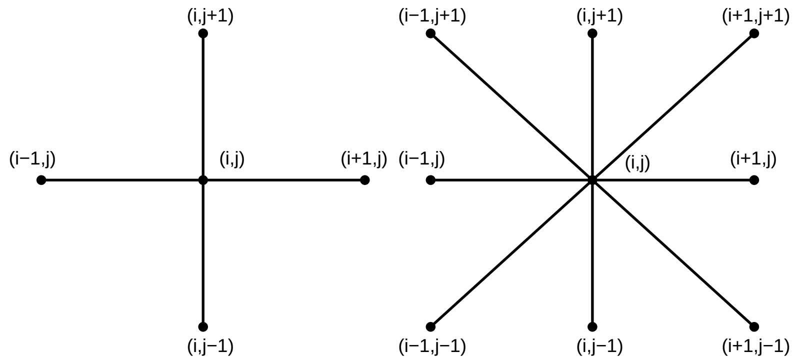

Figure 1: 2d DTV neighborhoods: Left, N_α^4. Right: N_α^8

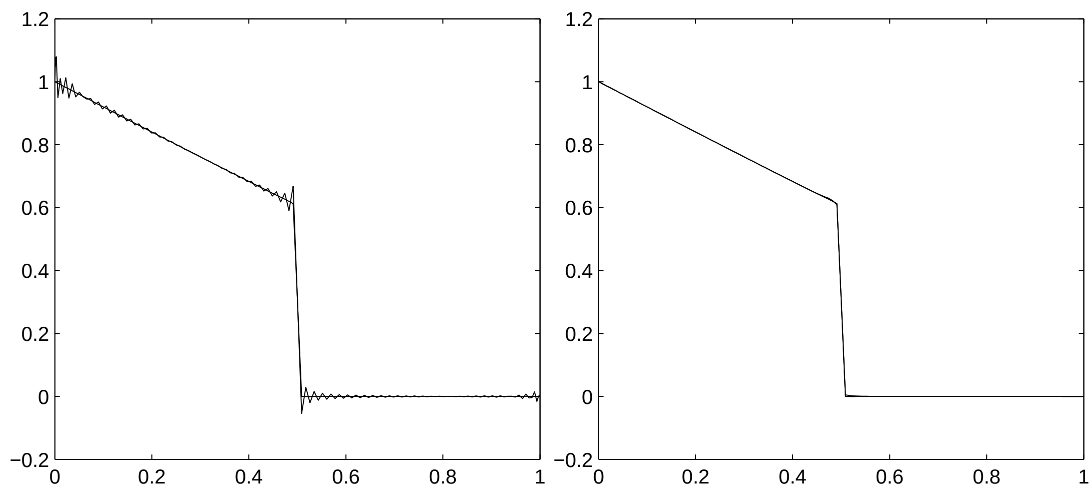

Figure 2: Left: Chebyshev pseudospectral (oscillatory) temperature vs. exact at $t = 0.5$. Right: DTV postprocessed temperature and the visually indistinguishable exact solution.

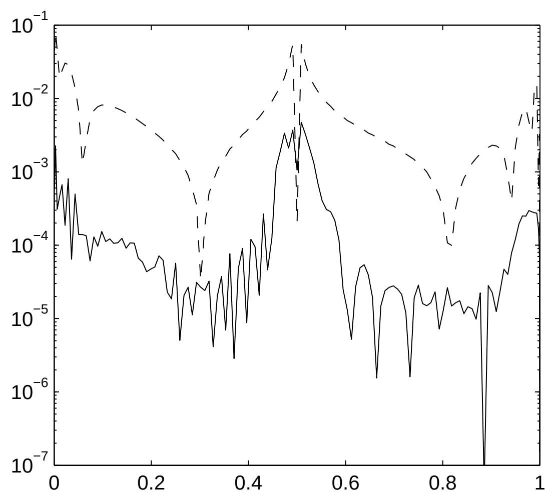

Figure 3: Approximation errors from the 1d problem in figure 2: Chebyshev pseudospectral (upper dashed) and DTV postprocessed (lower solid).

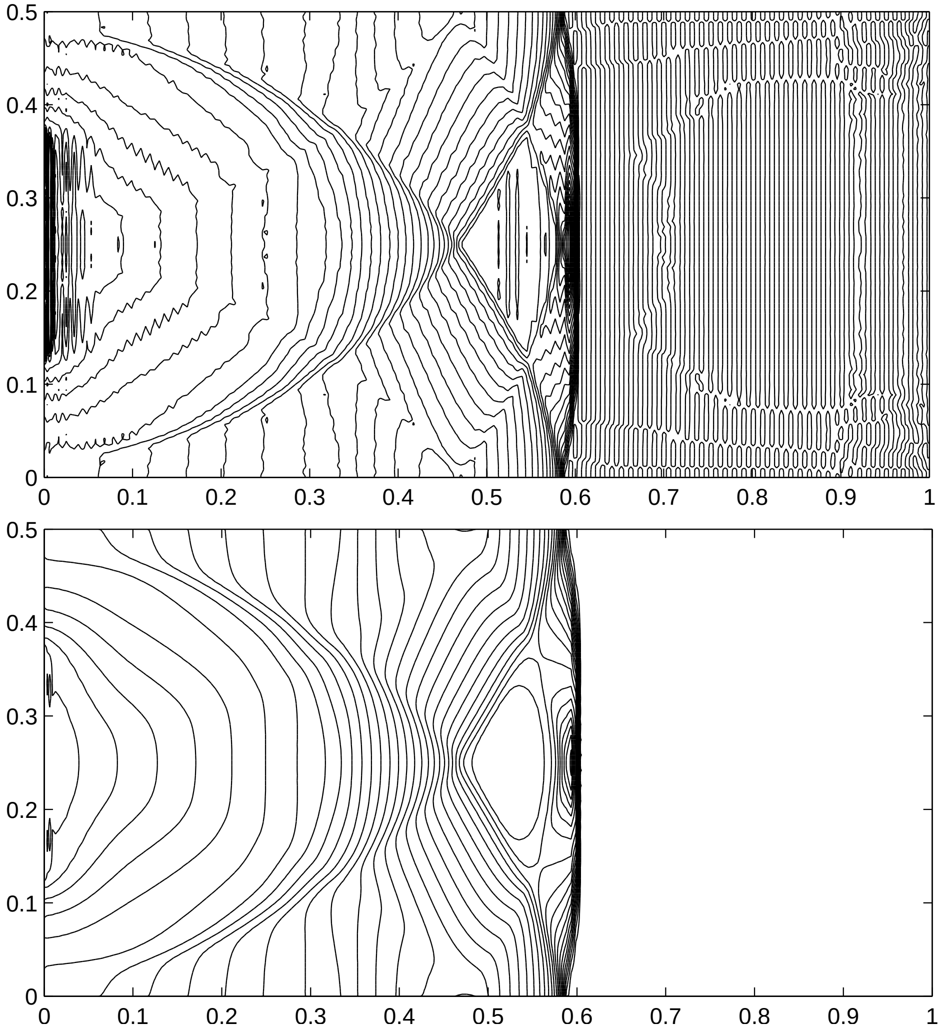

Figure 4: Contour plots with a constant interval of 0.025 for the first two-dimensional example. Upper: Chebyshev pseudospectral temperature at $t = 0.6$. Lower: DTV postprocessed temperature, $\lambda = 40$ and 200 time steps

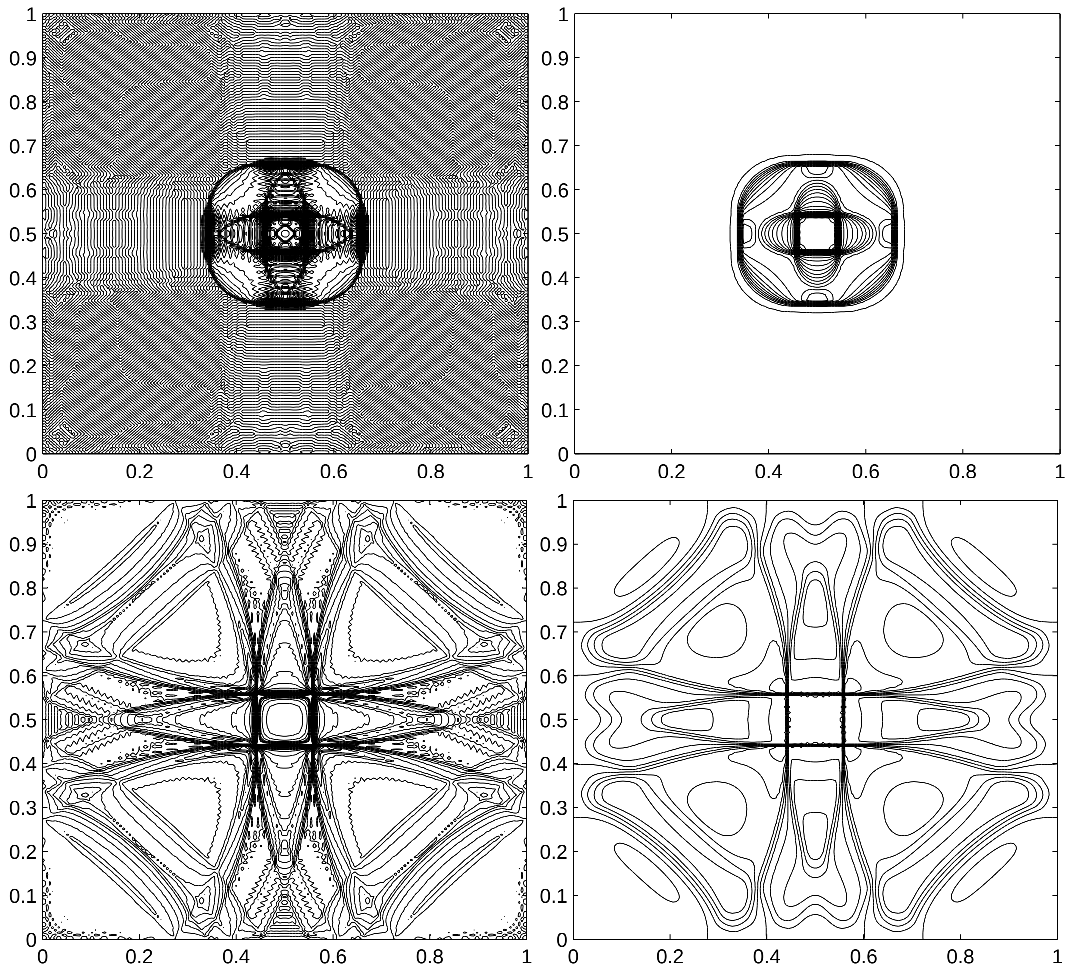

Figure 5: Upper: Temperature contour plots at $t = 0.1$ with a constant interval of 0.025 for the second two-dimensional example. Left: Chebyshev pseudospectral. Right: DTV postprocessed, $\lambda = 15$ and 200 time steps. Lower: Temperature at $t = 1.0$ with a constant interval of 0.01. Left: Chebyshev pseudospectral. Right: DTV postprocessed, $\lambda = 15$ and 200 time steps.